Variations on a theme: The sum of equal powers of natural numbers (II)

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This is the second of a series of notes, organized around the problem of finding a closed form for :

$$S_p(n) := \sum_{k=1}^{n} k^p = 1^p + 2^p + ... + n^p \text{ where } p, n \in \mathbb{N}.$$

In the first article, which appeared in issue 8 of this volume, we showed that $S_p(n)$ is always a polynomial in n of degree $p+1$, and we went over various ways to find that polynomial. As a review, the reader is encouraged to use one or more of these techniques to verify that

$$\begin{aligned} S_2(n) &= \frac{n(n+1)(2n+1)}{6} = S_1(n) \cdot \frac{2n+1}{3}, \\ S_3(n) &= \frac{n^2(n+1)^2}{4} = S_1^2(n), \\ S_4(n) &= \frac{n(n+1)(2n+1)\left(3n^2+3n-1\right)}{30} = S_2(n) \cdot \frac{6S_1(n)-1}{5}, \\ S_5(n) &= \frac{n^2(n+1)^2}{12}\left(2n^2+2n-1\right) = S_1^2(n) \cdot \frac{4S_1(n)-1}{3}. \end{aligned}$$

The frequency with which the factor $S_1(n)$ appears is striking. In what follows, we will investigate the reasons for this.

The structure of $S_p(n)$

Because $S_1(n)$ appears so often, we will abbreviate it as S. We thus have (see the article by V. S. Abramovich in issue 6 of this volume):

$$\begin{aligned} S_2(n) &= S \cdot \frac{2n+1}{3} \\ S_3(n) &= S^2 \\ S_4(n) &= S_2(n) \cdot \frac{6S-1}{5} = \frac{S(6S-1)}{5} \cdot \frac{2n+1}{3} \\ S_5(n) &= S^2 \cdot \frac{4S-1}{3} = S_3 \cdot \frac{4S-1}{3} \end{aligned}$$

We observe that $S_4(n) = P(S)S_2(n)$ and $S_5(n) = Q(S)S_3(n)$, where P and Q are first-degree polynomials with rational coefficients. Does this pattern continue? We conjecture that $S_p(n) = S^{\delta(p)} Q_p(S) \cdot M_p(n)$, where

$$M_p(n) = \begin{cases} 1 & \text{if } p \text{ is odd,} \\ \frac{2n+1}{3} & \text{if } p \text{ is even,} \end{cases} \qquad \delta(p) = \begin{cases} 2 & \text{if } p \text{ is odd,} \\ 1 & \text{if } p \text{ is even,} \end{cases}$$

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and $Q_p(S)$ is a polynomial of degree $\left[\frac{p+1}{2}\right]-\delta(p)$ with rational coefficients. Because we are looking at patterns that skip a value of p, it will be convenient to find recurrences that do the same thing – using $S_1, S_3, \ldots, S_{2n-1}$ to find S_{2n+1} and $S_2, S_4, \ldots, S_{2n-2}$ to find S_{2n}.

Recurrence relations for S_p for odd p and for even p

Exercise 1 *Let $p \geq 3$ be odd. Expand $(t+1)^{p+2}+(t-1)^{p+2}$ and show that this is equal to*

$$2\sum_{i=0}^{\frac{p+1}{2}}\binom{p+2}{2i}t^{p+2-2i}\,.$$

Use this to show that

$$S_p\left(n\right)=\frac{\left(n+1\right)^{p+2}-n^{p+2}-1-\left(p+2\right)\left(n^2+n\right)-2\sum_{i=2}^{\frac{p-1}{2}}\binom{p+2}{2i}S_{p+2-2i}\left(n\right)}{\left(p+2\right)\left(p+1\right)}\,. \tag{1}$$

Exercise 2 *Let $p \geq 2$ be even. Expand $(t+1)^{p+1}-(t-1)^{p+1}$ and show that this is equal to*

$$2\sum_{i=0}^{p/2}\binom{p+1}{2i+1}t^{p-2i}\,.$$

Use this to show that

$$S_p\left(n\right)=\frac{\left(n+1\right)^{p+1}+n^{p+1}-1-2\sum_{i=1}^{p/2}\binom{p+1}{2i+1}S_{p-2i}\left(n\right)}{2\left(p+1\right)}\,. \tag{2}$$

Exercise 3 *Using the above recursions or otherwise, prove that*

$$S_6\left(n\right)=\frac{n\left(n+1\right)\left(2n+1\right)\left(3n^4+6n^3-3n+1\right)}{42}=\frac{S_2\left(n\right)\left(6S^2-6S+1\right)}{7}$$

and further that

$$S_6(n)=\frac{S(6S^2-6S+1)}{7}\cdot\frac{2n+1}{3}$$

and

$$S_7\left(n\right)=S^2\cdot\frac{6S^2-4S+1}{3}.$$

So we have sufficient grounds to formulate two hypotheses (these are problems 4 and 5 in the Abramovich article):

a) For any odd $p \geq 3$, the quotient $\frac{S_p(n)}{S_3}=\frac{S_p(n)}{S^2}$ is a polynomial in S with rational coefficients;

b) For any even $p \geq 2$, the quotient $\frac{S_p(n)}{S_2(n)} = \frac{3S_p(n)}{S\cdot(2n+1)}$ is a polynomial in S with rational coefficients. (Or, using the above notation, prove that the quotient $\frac{S_p(n)}{S^{\delta(p)}M_p(n)}$ is a polynomial in S with rational coefficients.)

We will prove both hypotheses in an upcoming article, but for now we note one interesting property of $S_p\left(n\right)$ when p is odd. For any odd p, and any natural number n, the natural number $S_p\left(n\right)$ is divisible by the natural number $S_1\left(n\right).$ Indeed, upon reordering the terms we see that

$$S_p\left(n\right) = \sum_{k=1}^{n} k^p = \sum_{k=0}^{n-1} \left(n-k\right)^p . \qquad (3)$$

Note that $a^p + b^p$ is divisible by $a + b$ if p is odd since $a^p + b^p = a^p - (-b)^p$ and $a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + \cdots + b^{n-1})$. Then $k^p + (n-k)^p$ is divisible by $k + n - k = n$ for any $k = 1, 2, \ldots, n-1$ and therefore

$$\begin{aligned} 2S_p\left(n\right) = \sum_{k=1}^{n} k^p + \sum_{k=0}^{n-1} \left(n-k\right)^p &= \sum_{k=1}^{n-1} \left(k^p + (n-k)^p\right) + 2n^p \\ &= \sum_{k=1}^{n-1} \left(k^p + \sum_{i=0}^{p} \binom{p}{i} n^i (-k)^{p-i} \right) + 2n^p . \end{aligned}$$

As p is odd, we have

$$2S_p(n) = \sum_{k=1}^{n-1} \sum_{i=1}^{p} \left(\binom{p}{i} n^i (-k)^{p-i} \right) + 2n^p$$

and every term of the right-hand side is divisible by n.

Exercise 4 *By reindexing the right-hand side of (3), show that $(n+1)|2S_p(n)$. Conclude that the natural number $S_1(n)$ divides the natural number $S_p(n)$.*

Remark 1 *It is important to understand the subtlety of this statement that we just proved! Despite the fact that the coefficients of the polynomials $S_p(n)$ and $S_1(n)$ are rational numbers, their values and the value of the quotient $\frac{S_p(n)}{S_1(n)}$ are integers for any integer n. But that cannot be said about the quotient $\frac{S_p(n)}{S_1^2(n)}$ (find examples for which this quotient is not an integer). In fact, this quotient for odd $p \geq 3$ is also a polynomial with rational coefficients: this was confirmed above for $p = 3, 5, 7$ and will be proven for all p later.*

To prove hypothesis a) above, we need the following result.

Lemma 1 *For any odd $n \geq 5$, there are polynomials $K_n(t)$ with integer coefficients such that*

$$(x+1)^n - x^n - 1 - n(x^2+x) = (x^2+x)^2 K_n(x^2+x).$$

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Proof sketch. Let $t = x^2 + x$ and let $L_n = (x+1)^n - x^n$, $n \in \mathbb{N} \cup \{0\}$. Then $K_n = \frac{(x+1)^n - x^n - 1 - nt}{t}$ and $L_n = t^2 K_n + nt + 1$. We will prove that for any odd $n \geq 5$, K_n is a polynomial in t with integer coefficients.

Note that L_n can be defined by the recurrence

$$L_n = L_{n+2} - (2x+1)L_{n+1} + tL_n$$

for $n \in \mathbb{N} \cup \{0\}$, with $L_0 = 0$ and $L_1 = 1$ (check this!). Then substituting $L_n = t^2 K_n + nt + 1$ into this recurrence yields the recurrence

$$K_{n+4} = (2t+1)K_{n+2} - t^2 K_n - (n-2)t + 3$$

for odd $n \geq 5$ with $K_5 = 5, K_7 = 7(t+2)$. Therefore, K_n is a polynomial in t with integer coefficients. □

Exercise 5 *Using the lemma above and recurrence (1) for $S_p(n)$ for odd p, prove hypothesis a), that is that for any odd $p \geq 3$ there is a polynomial $Q_p(x)$ with rational coefficients such that $S_p(n) = S^2 \cdot Q_p(S)$; find a recursion for $Q_p(S)$.*

Remark 2 *Polynomials $S^2 \cdot Q_p(S)$, which equal to $S_p(n)$ for odd $p \geq 3$ are called* Faulhaber's polynomials *in honour of the German mathematician Johann Faulhaber (1580–1635) who first discovered this representation of $S_p(n)$ and computed the first seventeen of the polynomials. Recurrence (3) for calculating $Q_p(S)$ can be practically considered as a recurrence for Faulhaber's polynomials.*

Exercise 6 *Using recurrence (2) for $S_p(n)$ for even p, prove hypothesis b).*

References

Abramovich, V. S., *Sums of equal powers of natural numbers*, ***Crux* 40 (6)**, p. 248–252.

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